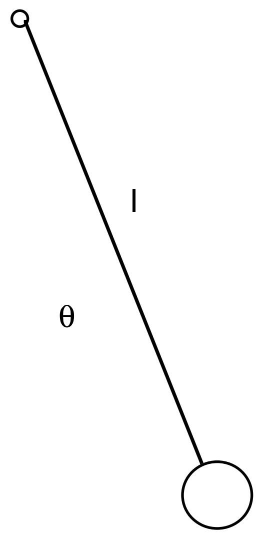
Assignment 6: Integration and the Simple Pendulum

We have already studied how the period of a pendulum is dependent on the amplitude when the amplitude becomes large. As we discussed in class, it is possible to use energy methods to find the dependence via integration rather than solving a differential equation with our Runge Kutta. This problem shows how we can use conservation of energy to solve an interesting problem in a different way.



We begin by writing the total energy

$$E = T + V$$
$$= \frac{1}{2}mv^2 + mgh$$

In this problem, we can write the linear velocity in terms of the angular velocity and the length of the pendulum. For convenience, we choose zero height to be at the pendulum pivot point.

$$v = l\frac{d\theta}{dt}$$
$$h = -l\cos\theta$$

This allows us to write the conservation of energy as

$$E = T + V$$
$$E = \frac{1}{2}m(l\frac{d\theta}{dt})^2 - mgl\cos\theta$$

We can invert this equation to solve for the angular velocity

$$E = \frac{1}{2}m(l\frac{d\theta}{dt})^2 - mgl\cos\theta$$
$$\frac{d\theta}{dt} = \sqrt{\frac{2E}{ml^2} + 2\frac{g}{l}\cos\theta}$$

We can now set up an integral that goes from a negative maximum amplitude to a positive and that can be related to half a period.

$$\frac{d\theta}{dt} = \sqrt{\frac{2E}{ml^2} + 2\frac{g}{l}\cos\theta}$$

$$\frac{d\theta}{\sqrt{\frac{2E}{ml^2} + 2\frac{g}{l}\cos\theta}} = dt$$

$$\int_0^{T/2} dt = \int_{-\theta_{max}}^{\theta_{max}} \frac{d\theta}{\sqrt{\frac{2E}{ml^2} + 2\frac{g}{l}\cos\theta}}$$

$$T = 2\int_{-\theta_{max}}^{\theta_{max}} \frac{d\theta}{\sqrt{\frac{2E}{ml^2} + 2\frac{g}{l}\cos\theta}}$$

The final step is to note that at the starting point (maximum amplitude) all of the energy is potential energy. This allows us to write the total energy in a nice way and then substitute back i.

$$E = \frac{1}{2}m(l\frac{d\theta}{dt})^2 - mgl\cos\theta$$

$$E = -mgl\cos\theta_{max}$$

$$T = 2\int_{-\theta_{max}}^{\theta_{max}} \frac{d\theta}{\sqrt{\frac{2E}{ml^2} + 2\frac{g}{l}\cos\theta}}$$

$$= 2\int_{-\theta_{max}}^{\theta_{max}} \frac{d\theta}{\sqrt{\frac{2(-mgl\cos\theta_{max})}{ml^2} + 2\frac{g}{l}\cos\theta}}$$

$$= 2\int_{-\theta_{max}}^{\theta_{max}} \frac{d\theta}{\sqrt{-2\frac{g}{l}\cos\theta_{max} + 2\frac{g}{l}\cos\theta}}$$

$$= \frac{2}{\sqrt{2}}\cdot\sqrt{\frac{l}{g}}\cdot\int_{-\theta_{max}}^{\theta_{max}} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_{max}}}$$

We are left with an integral that cannot be written in terms of simple algebraic or trigonometric functions. It is known as as an "elliptic integral" and it is an excellent candidate for numerical integration.

Program

Write a program (or programs) to find the period by integrating the elliptic integral shown above for initial positions that range from 0.1 radians to 3.14 radians using both simple integration and Gaussian Quadrature.

How to do it.

You will need to read in the initial amplitude and the length of the pendulum and the parameters for integration (initial angle and number of p from either the user's terminal or from a file. For simple integration, it is best to read in the number of iterations and calculate the step size in theta. For Gaussian quadrature, you will need to read the number of Gauss points.

You should use the code that I sent to you for acquiring the Gauss points. It will provide up to 96 points. You will need to embed some kind of mapping in your quadrature program so that the region is mapped correctly.

Suggestion: Write two separate programs--one that uses the simple integration and one that uses the Gaussian Quadrature. While these are separate programs, structure them so that they can read the same input file and use the same "f" to plug in to integrate.